A gas dynamics scheme for a two moments model of radiative transfer

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Abstract

We address the discretization of the Levermore's two moments and entropy model of the radiative transfer equation. We present a new approach for the discretization of this model: first we rewrite the moment equations as a Compressible Gas Dynamics equation by introducing an additionnal quantity that plays the role of a density. After that we discretize using a Lagrange-projection scheme. The Lagrange-projection scheme permits us to incorporate the source terms in the fluxes of an acoustic solver in the Lagrange step, using the well-known "piecewise steady approximation" and thus to capture correctly the diffusion regime. Moreover we show that the discretization is entropic and preserve the flux-limited property of the moment model. Numerical examples illustrate the feasability of our approach.

Contents

1 Introduction

We are interested in the discretization of the equations of radiative transfer by means of accurate and stable coarse grid techniques. In this direction we have studied, in [1], an implicit discretization of a radiative two moments model based on relaxation plus well-balanced scheme in dimension one. The discrete model has two small parameters. The first one ε is a physical parameter that measures the ratio of the sound velocity over the light velocity. The second parameter Δx is the mesh size. The direct eulerian method [1] is very well adapted to one dimensional problems and consequently to multidimensional problems computed on a cartesian mesh. We refer to [5] for an extension on a cartesian 2D mesh in the context of the explicit HLLC solver. In this work we address a new problem, which consists in discretization techniques that can be used on a general non structured mesh in dimension greater than one. The previous method is not able to answer this question because it needs the solution of some steady state problems. The solution of these steady state problems is possible on a cartesian mesh in dimension one, but as far as we know, no general solution exists on a general multidimensional mesh in the context of direct explicit or implicit eulerian methods. Therefore we do not know how to respect the diffusion limit $\varepsilon \to 0$ of the model on non structured coarse grid with the method [1]. This is why we explore a new method. Since the ideas behind this method seem to be new, we evaluate it in dimension one. However the scheme that we propose could be easily extended in dimension greater than one.

The starting point is the observation [12] that moment models for anisotropic flows are isotropic if we rewrite the equations in an appropriate moving frame. This statement is true at the level of principles. We use this idea in our work by showing the Levermore moments model based on the entropy closure may be recast as a classical gas dynamics system using new unknowns. Since this equivalence is very important in this work, we propose to call such systems GDL for Gas Dynamics Like. As a consequence the radiative pressure which is a tensor is splitted in a scalar pressure and displacement. The scalar pressure is the isotropic pressure of radiation in the frame attached with the radiation.

Therefore it gives us the opportunity to solve our problem with some standard two steps Lagrange plus projection methods which are classicaly used for the numerical solution of gas dynamic equations. We study and evaluate one of them. We show that the new scheme has two properties. First the scheme is entropy increasing and therefore is flux limited in the sense that the modulus of the radiation flux is always smaller than the radiation energy. The technique of the proof seems to be more powerful than the previous one [1, 5] since we treat an radiative flux with two components while the previous published results concerned only a radiative flux with one component. Second the diffusion limit of the scheme is correct. Our proof is in the weak sense while the classical one [6] relies on strong convergence. To get this property, we have generalized in our lagrangian framework the steady-state approximation scheme, see [6]. The construction of this new steady state approximation is much easier in the lagrangian two steps framework used in this approach than in the more traditional eulerian and direct framework that was used in [1]. We think the two steps lagrange plus projection scheme is the key point for the construction of this asymptotic preserving scheme. The new approach is more adequate for a multi-dimensional extension. We illustrate the correctness of this method on a simple test case and show the diffusion limit is well captured on a coarse grid.

This work is organized as follows. In section 2 we rewrite the moments model as a Gas Dynamics Like system with an appropriate definition of the pressure. In section 3 we recall the streaming regime and the diffusion regime. In section 4 we show there exists other Gas Dynamics Like systems. Section 5 is devoted to the construction of a numerical scheme. In section 6 we show the positivity of the scheme. In section 7 we prove the diffusion limit is correct. Finally we give the result of one numerical experiment in section 8.

In some parts of the paper we will simplify the notations using $\varepsilon = 1$. The numerical tests have been done with $\varepsilon \ll 1$.

2 Derivation of the equations

Let consider the M^1 moment moment model for radiative hydrodynamics in dimension $d = 1, 2, 3$

$$\left\{ \begin{array}{l} \partial_t E + \frac{1}{\varepsilon}\nabla.F = 0, \\ \partial_t F + \frac{1}{\varepsilon}\nabla.P = 0. \end{array} \right. \tag{1}$$

We consider the entropy closure of [7] for which one can parametrize the radiative energy $E \in \mathbb{R}$, flux $F \in \mathbb{R}^d$ and pressure $P \in \mathbb{R}^{d\times d}$ by

$$\left\{ \begin{array}{l} E = \frac{3+|b|^2}{3(1-|b|^2)^3} T^4, \\ F = -\frac{4b}{3(1-|b|^2)^3} T^4, \\ P = \left(\frac{1-\chi}{2} I + \frac{3\chi - 1}{2} \frac{f\otimes f}{|f|^2} \right) E, \end{array} \right. \tag{2}$$

where $f = \frac{F}{E}$ is the non dimensional radiation flux and

$$\chi = \frac{3 + 4|f|^2}{5 + 2\sqrt{4 - 3|f|^2}} \tag{3}$$

is the Eddington factor. This closure model is compatible with an entropy-entropy flux pair

$$S = \frac{4}{3(1-|b|^2)^2} T^3 \in \mathbb{R}, \quad Q = -bS \in \mathbb{R}^d. \tag{4}$$

It means that smooth solutions satisfy $\partial_t S + \frac{1}{\varepsilon}\nabla.Q = 0$ while discontinuous solutions satisfy $\partial_t S + \frac{1}{\varepsilon}\nabla.Q \geq 0$ in the weak sense (S is the physical concave entropy).

The starting point of our analysis is a quite strong formal analogy between this system and compressible gas dynamics. Let us define what we will refer to as the velocity

$$u = -b \in \mathbb{R}^d \tag{5}$$

and a density $\rho \in \mathbb{R}$

$$\partial_t \rho + \frac{1}{\varepsilon}\nabla.(\rho u) = 0.$$

The density depends of course of some artificial initial condition which we do not discuss for the moment. Let us define also a scalar

$$q = \frac{1}{3(1-|b|^2)^2} T^4 \in \mathbb{R} \tag{6}$$

Lemma 1. *On has the relations*

$$F = uE + qu \text{ and } P = u \otimes F + qI. \tag{7}$$

The first relation comes from

$$uE + qu = -b(E+q) = -b\frac{(3+|b|^2)+(1-|b|^2)}{3(1-|b|^2)^3}T^4 = -\frac{4b}{3(1-|b|^2)^3}T^4 = F.$$

It remains to prove the second relation. Let us check that

$$\chi = \frac{1+3|b|^2}{3+|b|^2}. \tag{8}$$

One has

$$\chi = \frac{3+4|f|^2}{5+2\sqrt{4-3|f|^2}} = \frac{5-2\sqrt{4-3|f|^2}}{3}. \tag{9}$$

From

$$f = \frac{F}{E} = -\frac{4b}{3+|b|^2} \tag{10}$$

one gets

$$4 - 3|f|^2 = 4\frac{(3-|b|^2)^2}{(3+|b|^2)^2}. \tag{11}$$

Therefore plugging in (9) one gets (8). It is then an easy task to check the second relation of (7). One has separately

$$\frac{1-\chi}{2}E = \frac{1}{3(1-|b|^2)^2}T^4 = q,$$

and

$$\frac{3\chi-1}{2}\frac{f\otimes f}{|f|^2}E = \frac{3\chi-1}{2}\frac{b\otimes b}{|b|^2}E$$

$$= 4\frac{|b|^2}{3+|b|^2}\frac{b\otimes b}{|b|^2}\frac{3+|b|^2}{3(1-|b|^2)^3}T^4 = \frac{4b\otimes b}{3(1-|b|^2)^3}T^4.$$

Adding these, one gets $P = qI + u\otimes F$. The relation is proved.

Using these relations we can rewrite the equation of radiation as a system which is formally close to the standard system of gas dynamics

$$\left\{\begin{array}{l} \partial_t \rho + \frac{1}{\varepsilon}\nabla.(\rho u) = 0, \\ \partial_t \rho v + \frac{1}{\varepsilon}\nabla.(\rho u \otimes v) + \frac{1}{\varepsilon}\nabla q = 0, \\ \partial_t \rho e + \frac{1}{\varepsilon}\nabla.(\rho u e + q u) = 0, \\ \partial_r \rho s + \frac{1}{\varepsilon}\nabla.(\rho u s) \geq 0, \end{array}\right. \tag{12}$$

where by definition

$$S = \rho s, \quad F = \rho v \text{ and } E = \rho e. \tag{13}$$

q can be computed directly with respect to the main unknowns of this system

$$q = \frac{1 - |b|^2}{3 + |b|^2}\rho e. \tag{14}$$

From (10-11) one gets

$$b = -\frac{3f}{2 + \sqrt{4 - 3|f|^2}}.$$

Since $f = \frac{F}{E} = \frac{\rho|v|}{\rho e} = \frac{|v|}{e}$ then q is a function of v and e. Therefore the system is closed and the scalar pressure is non singular.

The vector v is different from the "velocity" u. But they are nevertheless colinear since

$$v = \frac{4T^4}{3(1 - |b|^2)^3\rho}u.$$

Due to the analogy between this system and the system of compressible gas dynamics we propose to call such a system Gas Dynamics Like (GDL in the rest of the paper). Our purpose is to evaluate some consequence of this analogy. A first consequence is that the total radiative pressure is now splitted into a convective part and a isotropic part, like in classical gas dynamics.

3 Different regimes

Two regimes are important for radiative flows, on one side the streaming regime and on the other side the diffusion regime. We now analyse these regimes.

3.1 Free streaming

Free streaming is free radiation flow. Assume that

$$\frac{|F|}{E} = 1$$

everywhere at $t = 0$. Then $|f| = 1$ and so $|b| = 1$. In this regime some of the equations are singular. At least the mapping $(E, F) \mapsto (T, b)$ is singular. However system (1) together with (2) is well defined because the total pressure

P is a well defined function of E and F even in this regime. Similarly system (12-14) is well defined. In the streaming regime one has

$$q = 0$$

everywhere at $t = 0$. Since f is parallel to b and therefore parallel to u, then the system is equivalent to the so-called pressureless gas dynamic system. Therefore it inherits, in this regime, the properties of pressureless gas dynamics system: only weak hyperbolicity, exact propagation along the rays until some non linear interaction appears. It proves

Property 2. *Assume the solution in dimension 3 at $t = 0$ is smooth and such that $\frac{|F|}{E} = 1 \Longleftrightarrow q = 0$. Let $\Omega \in \mathbb{R}^3$ be such that $|\Omega| = 1$ be a direction variable. Consider $I(t, x; \Omega)$ the solution of*

$$\partial_t I + \frac{1}{\varepsilon}\Omega.\nabla_x I = 0, \quad I(t = 0, x; \Omega) = E(0, x)\delta_{\Omega - \frac{F(0,x)}{E(0,x)}}.$$

There exists a time $T > 0$ such that

$$E(t, x) = \frac{\int_{|\Omega|=1} I(t, x; \Omega)d\Omega}{\int_{|\Omega|=1} d\Omega}, \quad F(t, x) = \frac{\int_{|\Omega|=1} I(t, x; \Omega)\Omega d\Omega}{\int_{|\Omega|=1} d\Omega} \text{ and } \frac{|F|}{E} = 1 \Longleftrightarrow q = 0.$$

.

We deduce that for such prepared initial data, the M^1 moment model is equivalent to the transport equation.

3.2 Diffusion regime

It is the asymptotic regime of the equations with a stiff right hand side. Consider

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\nabla.(\rho u) = 0, \\ \partial_t \rho v + \frac{1}{\varepsilon}\nabla.(\rho u \otimes v) + \frac{1}{\varepsilon}\nabla q = -\frac{\sigma}{\varepsilon^2}\rho v, \\ \partial_t \rho e + \frac{1}{\varepsilon}\nabla.(\rho u e + q u) = -(T^4 - \rho e). \end{cases} \tag{15}$$

The asymptotic regime is a consequence of

$$\rho v = -\frac{\varepsilon}{\sigma}\left(\nabla.(\rho u \otimes v) + \nabla q\right) + o(\varepsilon)$$

which implies that ρv is $O(\varepsilon)$. Therefore in this regime the convective part $\nabla.(\rho u \otimes v)$ is $O(\varepsilon^2)$. Only the scalar pressure contributes

$$\rho v = -\frac{\varepsilon}{\sigma}\nabla q + o(\varepsilon). \tag{16}$$

By construction $\rho u e + q u = \rho v = F$.

Property 3. *The diffusion limit of (15) is*

$$\partial_t \rho e - \nabla.(\frac{1}{\sigma}\nabla q) = -(T^4 - \rho e), \quad q = \frac{\rho e}{3}.$$

4 Other GDL systems

In this section we consider a system in one dimension (for simplicity)

$$\begin{cases} \partial_t E + \partial_x F = 0, \\ \partial_t F + \partial_x P = 0. \end{cases} \tag{17}$$

where we have dropped the $\frac{1}{\varepsilon}$ for convenience. We would like to determine other models which are GDL. Assuming such a system is GDL means by comparison with (12) that it is possible to define a velocity u such that (17) can be rewritten as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t \rho v + \partial_x(\rho u v + q) = 0, \\ \partial_t \rho e + \partial_x(\rho u e + q u) = 0 \end{cases} \tag{18}$$

for well chosen unknowns. It is here understood that

$$\rho e = E \text{ and } \rho v = F.$$

We also add that the entropy of the system satisfies (for smooth solutions)

$$\partial_t \rho s + \partial_x(\rho u s) = 0.$$

Set $S = \rho s$ and $Q = \rho u s$. Since by hypothesis S is the entropy of (17) there exists a and b such that

$$dS = a(dE + bdF) \text{ and } dQ = a(dF + bdP). \tag{19}$$

The system (17) is GDL if and only if the following condition is fulfilled

$$P = uF + q \tag{20}$$

where q is defined by

$$q = \frac{F}{u} - E.$$

Lemma 4. *If the system is GDL then*

$$q = \frac{S}{a} - E - bF. \tag{21}$$

The result is easily proved using the material derivative

$$D_t = \partial_t + u\partial_x.$$

The entropy equation becomes $\rho D_t s = 0$. From (19) one has

$$ds = a(de + bdf - (\frac{S}{a} - (E + bF)d\tau)), \quad \tau = \frac{1}{\rho}.$$

So

$$0 = \rho D_t s = a(\rho D_t e + b\rho D_t f - (\frac{S}{a} - (E + bF)\rho D_t \tau))$$

$$a(-\partial_x(qu) - b\partial_q + (\frac{S}{a} - (E+bF)\partial_x u))$$
$$= a((\frac{S}{a} - (E+bF) - q)\partial_x u - (b+u)\partial_x q.$$

Since u and q are arbitrary, it means that

$$b + u = 0 \text{ and } \frac{S}{a} - (E+bF) - q = 0.$$

The proof is ended.

We can use this expression in the definition of q. It gives $\frac{F}{u} - E = \frac{S}{a} - (E+bF)$ so

$$S = -a(1-b^2)\frac{F}{b}. \tag{22}$$

By inspection of (2), we add one more hypothesis. We assume there exist some functions $\varphi(b)$, $\psi(b)$ $h(b)$ and a real number $\alpha \in \mathbb{R}$ such that

$$\begin{cases} E = a^\alpha \varphi(b), \\ F = a^\alpha \psi(b), \\ S = a^{\alpha+1} h(b). \end{cases} \tag{23}$$

Plugging in $dS = a(de + bdF)$ it gives a compatibility condition due to

$$(\alpha+1)a^\alpha h(b)da + a^{\alpha+1}h'(b)db$$
$$= a\left(\alpha a^{\alpha-1}\varphi(b)da + a^\alpha \varphi'(b)db + \alpha a^{\alpha-1} b\psi(b)da + a^\alpha b\psi'(b)db\right).$$

Equating the terms in front of da and db one gets

$$(\alpha+1)h(b) = \alpha(\varphi(b) + b\psi(b))$$

and

$$h'(b) = \varphi'(b) + b\psi'(b).$$

So $\varphi(b) = \frac{\alpha+1}{\alpha}h(b) - b\psi(b)$ and

$$\varphi'(b) = \frac{\alpha+1}{\alpha}h'(b) - b\psi'(b) - \psi(b).$$

So

$$h'(b) = \frac{\alpha+1}{\alpha}h'(b) - \psi(b) \Longleftrightarrow h'(b) = \alpha\psi(b).$$

On the other hand (22) implies

$$\psi(b) = -\frac{b}{1-b^2}h(b).$$

Therefore we obtain an ordinary differential equation

$$h'(b) = -\alpha\frac{b}{1-b^2}h(b). \tag{24}$$

We solve easily

$$\frac{h'}{h} = \frac{\alpha}{2}\frac{2b}{b^2-1}$$

so

$$h(b) = \frac{\alpha}{\alpha+1}|1-b^2|^{\frac{\alpha}{2}}.$$

The constant $\frac{\alpha}{\alpha+1}$ is here only for compatibility with the M^1 equations. A singularity is present for $b=1$. Since b is in practice a measure of the anisotropy, it means that we are more interested by the range $|b| \leq 1$. In summary we have proved

Lemma 5. *All 1D systems such that*

$$\left\{ \begin{array}{l} S = \frac{\alpha}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}} a^{\alpha+1} a^{\alpha+1}, \\ E = \frac{(\alpha+1)-b^2}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}-1} a^{\alpha+1} \\ F = -\frac{\alpha}{\alpha+1} b(1-b^2)^{\frac{\alpha}{2}-1} a^{\alpha+1}, \\ q = -\frac{1}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}} a^{\alpha+1}, \\ P = \frac{(\alpha+1)b^2-1}{\alpha+1}(1-b^2)^{\frac{\alpha}{2}} a^{\alpha+1}, \end{array} \right. \tag{25}$$

are GDL for $|b| \leq 1$.

5 Numerical methods

Many numerical schemes exist for compressible gas dynamics. We consider one among our favorite, and generalize it to (12). This scheme is split in two stages, one lagrangian stage and one remapping stage. We will also show that discrete radiative quantities do not depend on the particular value of the (artificial) discrete density.

5.1 Lagrangian step

First we rewrite the system in quasi-lagrangian coordinates

$$\left\{ \begin{array}{l} \rho D_t \tau + \partial_x b = 0, \\ \rho D_t v + \partial_x q = 0, \\ \rho D_t e - \partial_x (qb) = 0. \end{array} \right.$$

The material derivative is defined by

$$D_t = \partial_t + u\partial_x.$$

Second we linearize it considering the entropy is constant

$$\rho D_t s = 0$$

which is true for smooth solutions. So we consider the isentropic system

$$\left\{ \begin{array}{l} \rho D_t \tau + \partial_x b = 0, \\ \rho D_t v + \partial_x q = 0. \end{array} \right.$$

Next we need to compute the Riemann invariant. The fundamental principle of thermodynamics for our system writes

$$Tds = de + bdv + qd\tau. \tag{26}$$

Therefore

$$d(e + bv + q\tau) = Tds + vdb + \tau dq.$$

The definition of q implies

$$e + bv + q\tau = Ts = 3^{\frac{1}{4}} s q^{\frac{1}{4}} (1 - b^2)^{\frac{1}{2}}.$$

Therefore easy computations show

$$\frac{\partial(\tau, v)}{\partial(b, q)} = -\alpha \begin{pmatrix} \frac{1}{4} qb(1-b^2) & \frac{3}{16}(1-b^2)^2 \\ q^2 & \frac{1}{4} qb(1-b^2) \end{pmatrix}, \quad \alpha = 3^{\frac{1}{4}} s q^{-\frac{7}{4}} (1-b^2)^{-\frac{3}{2}}.$$

The eigenvalues of the matrix in parenthesis are

$$\lambda^- = \frac{1}{4} p(1-b^2)(b - \sqrt{3}), \quad \lambda^+ = \frac{1}{4} p(1-b^2)(b + \sqrt{3}).$$

The eigenvectors are

$$r^- = (q, -\frac{\sqrt{3}}{4}(1-b^2)), \quad r^+ = (q, \frac{\sqrt{3}}{4}(1-b^2)).$$

So we get

$$-\left(3^{\frac{1}{4}} s q^{-\frac{7}{4}} - (1-b^2)^{-\frac{3}{2}}\right) \lambda^+ \left(r^+, D_t(b,q)\right) + \left(r^+, \partial_x(b,q)\right) = 0$$

and

$$-\left(3^{\frac{1}{4}} s q^{-\frac{7}{4}} (1-b^2)^{-\frac{3}{2}}\right) \lambda^- \left(r^-, D_t(b,q)\right) + \left(r^-, \partial_x(b,q)\right) = 0.$$

An approximate method for the construction of the scheme consists in taking $(r^+, D_t(b,q))$ as a right Riemann invariant. Let us consider that a right state (b_R, q_R) is given. The intermediate state should satisfy

$$q_R(b^* - b_R) + \frac{\sqrt{3}}{4}(1 - b_R^2)(q^* - q_R) = 0.$$

Similarly from the left Riemann invariant we get

$$q_L(b^* - b_L) - \frac{\sqrt{3}}{4}(1 - b_L^2)(q^* - q_L) = 0.$$

These equations are equivalent to the linear system

$$\begin{cases} (q^* - q_R) + \frac{4}{\sqrt{3}} \frac{E_R}{3 + |b_R|^2}(b^* - b_R) = 0, \\ (q^* - q_L) - \frac{4}{\sqrt{3}} \frac{E_L}{3 + |b_L|^2}(b^* - b_L) = 0. \end{cases} \tag{27}$$

The solution is

$$\begin{cases} b^* = \dfrac{\frac{E_L}{3+|b_L|^2}b_L + \frac{E_R}{3+|b_R|^2}b_R}{\frac{E_L}{3+|b_L|^2} + \frac{E_R}{3+|b_R|^2}} + \dfrac{\sqrt{3}}{4}\dfrac{q_R - q_L}{\frac{E_L}{3+|b_L|^2} + \frac{E_R}{3+|b_R|^2}}, \\ q^* = \dfrac{\frac{3+|b_L|^2}{E_L}q_L + \frac{3+|b_R|^2}{E_R}q_R}{\frac{3+|b_L|^2}{E_L} + \frac{3+|b_R|^2}{E_R}} + \dfrac{4}{\sqrt{3}}\dfrac{b_R - b_L}{\frac{3+|b_L|^2}{E_L} + \frac{3+|b_R|^2}{E_R}}. \end{cases} \tag{28}$$

A standard Lagrangian scheme is now

$$\begin{cases} \Delta m_j^n \frac{\widetilde{\tau_j^{n+1}} - \tau_j^n}{\Delta t} + b^n_{j+\frac{1}{2}} - b^n_{j-\frac{1}{2}} = 0, \\ \Delta m_j^n \frac{\widetilde{v_j^{n+1}} - v_j^n}{\Delta t} + q^n_{j+\frac{1}{2}} - q^n_{j-\frac{1}{2}} = 0, \\ \\ \Delta m_j^n \frac{\widetilde{e_j^{n+1}} - e_j^n}{\Delta t} - q^n_{j+\frac{1}{2}} b^n_{j+\frac{1}{2}} + q^n_{j-\frac{1}{2}} b^n_{j-\frac{1}{2}} = 0, \end{cases} \tag{29}$$

where the fluxes are computed using (28) and where the mass of the cell is

$$\Delta m_j^n = \rho_j^n \Delta x = \frac{\Delta x}{\tau_j^n}. \tag{30}$$

The quantities after this lagrangian step will be modified after the remap step.

5.2 Discretization of the source term

The lagrangian system with a source term modeling the scattering of radiation is

$$\begin{cases} \rho D_t \tau + \partial_x b = 0, \\ \rho D_t v + \partial_x q = -\sigma \rho v, \\ \rho D_t e - \partial_x (qb) = 0. \end{cases} \tag{31}$$

This source term is important for the capture of the diffusion limit of scheme, in the stiff case (see system (15)). A first possibility for the discretization of the relaxation $-\sigma\rho v$ is to use a splitting strategy. That is: First one solves the Lagrangian system using (29); Second one discretizes the ordinary differential equation

$$\rho D_t v = -\sigma \rho v$$

in the cell during the time step. It is known this strategy fails to capture the diffusion limit for stiff problems. Therefore we do not recommend it.

The second possibility is to remark that the right hand side is analogous to a friction or a gravity right hand side in the equations of compressible gas dynamics. In such a case a possibility is to incorporate the right hand side in the definition of the fluxes such that the second equation in (31) is guaranteed by construction in the stationary case. So we replace (27) by

$$\begin{cases} (q^* - q_R) + \frac{4}{\sqrt{3}}\frac{E_R}{3+|b_R|^2}(b^* - b_R) = \frac{\sigma}{2}\Delta x \rho_R v_R, \\ (q^* - q_L) - \frac{4}{\sqrt{3}}\frac{E_L}{3+|b_L|^2}(b^* - b_L) = -\frac{\sigma}{2}\Delta x \rho_L v_L. \end{cases} \tag{32}$$

With respect to (27) q_R and q_L are just modified. The solution is

$$\left\{\begin{array}{l} b^* = \dfrac{\frac{E_L}{3+|b_L|^2}b_L + \frac{E_R}{3+|b_R|^2}b_R}{\frac{E_L}{3+|b_L|^2} + \frac{E_R}{3+|b_R|^2}} + \dfrac{\sqrt{3}}{4}\dfrac{\left(q_R + \frac{\sigma}{2}\Delta x \rho_R v_R\right) - \left(q_L - \frac{\sigma}{2}\Delta x \rho_L v_L\right)}{\frac{E_L}{3+|b_L|^2} + \frac{E_R}{3+|b_R|^2}}, \\ q^* = \dfrac{\frac{3+|b_L|^2}{E_L}\left(q_L - \frac{\sigma}{2}\Delta x \rho_L v_L\right) + \frac{3+|b_R|^2}{E_R}\left(q_R + \frac{\sigma}{2}\Delta x \rho_R v_R\right)}{\frac{3+|b_L|^2}{E_L} + \frac{3+|b_R|^2}{E_R}} + \dfrac{4}{\sqrt{3}}\dfrac{b_R - b_L}{\frac{3+|b_L|^2}{E_L} + \frac{3+|b_R|^2}{E_R}}. \end{array}\right. \tag{33}$$

Another choice could be to remark that

$$\rho v = -kb \text{ with } k = -\frac{T^4}{3(1-|b|^2)^3}.$$

So the right hand side in (32) becomes now

$$\left\{\begin{array}{l} (q^* - q_R) + \frac{4}{\sqrt{3}}\frac{E_R}{3+|b_R|^2}(b^* - b_R) = -\frac{\sigma}{2}\Delta x k_R b^*, \\ (q^* - q_L) - \frac{4}{\sqrt{3}}\frac{E_L}{3+|b_L|^2}(b^* - b_L) = \frac{\sigma}{2}\Delta x k_L b^*. \end{array}\right. \tag{34}$$

The solution of this linear system always exists.

5.3 Remap step

The remap step is standard for gas dynamics. Since the lagrangian step of the algorithm is equivalent to solving gas dynamics on a moving mesh, we just move the mesh. The velocity of the moving mesh is of course

$$\widetilde{x_{j+\frac{1}{2}}^{n+1}} = x_{j+\frac{1}{2}}^n + \Delta t u_{j+\frac{1}{2}}^n = x_{j+\frac{1}{2}}^n - \Delta t b_{j+\frac{1}{2}}^n.$$

A convenient notation is to note $\widetilde{x_j}$ the length of the cell after the displacement of the mesh

$$\widetilde{\Delta x_j} = \Delta x + \Delta t\left(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n\right) = \widetilde{x_{j+\frac{1}{2}}^{n+1}} - \widetilde{x_{j-\frac{1}{2}}^{n+1}}.$$

After that we project the numerical solution onto the old mesh in a conservative fashion.

5.4 Independence with respect to the density

The first equation of (29) combined with the definition (30) of the mass of the cell at the beginning of the time step is equivalent to

$$\widetilde{\rho_j^{n+1}}\widetilde{\Delta x_j} = \rho_j^n \Delta x.$$

Therefore the second equation of (29) is equivalent to

$$\frac{\widetilde{\Delta x_j}\widetilde{F_j^{n+1}} - \Delta x F_j^n}{\Delta t} + q_{j+\frac{1}{2}}^n - q_{j-\frac{1}{2}}^n = 0.$$

Similarly the third equation is equivalent to

$$\frac{\widetilde{\Delta x_j}\widetilde{E_j^{n+1}} - \Delta x E_j^n}{\Delta t} - q_{j+\frac{1}{2}}^n b_{j+\frac{1}{2}}^n + q_{j-\frac{1}{2}}^n v_{j-\frac{1}{2}}^n = 0.$$

The new length $\widetilde{\Delta x_j}$ is equal to the old one plus an increment which is a function of $b_{j+\frac{1}{2}}^n$ and $b_{j-\frac{1}{2}}^n$ which are by definition independent of the density at the beginning of the time step. Therefore $\widetilde{F_j^{n+1}}$ and $\widetilde{E_j^{n+1}}$ are independent of the definition the density at the beginning of the time step. The projection step is purely geometric and is therefore independent of the densities.

In case we use (33) or (34) the conclusions are the same.

6 Maximum principle

In this work maximum principle means

$$E \geq 0 \text{ and } |b| \leq 1 \Longleftrightarrow E \pm |F| \geq 0.$$

For the sake of the simplicity, we present the semi-discrete case and give only a sketch of the proof. In the sequel we show that the maximum principle can be viewed as a consequence of a standard entropy inequality.

6.1 Maximum principle for the Lagrangian system

The fully discrete entropy inequality is proved in [3, 4, 2]. The fully implicit case does not raise new theoretical issues. The proof highlights the connection between the moment model and gas dynamics through the fundamental principle of thermodynamics for the model

$$T_j(t)s_j'(t) = e_j'(t) + b_j(t)v_j'(t) + q_j(t)\tau_j'(t).$$

Consider the semi-discrete case with source term

$$\left\{\begin{array}{l} \Delta m_j \tau_j'(t) + b_{j+\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t) = 0, \\ \Delta m_j v_j'(t) + q_{j+\frac{1}{2}}(t) - q_{j-\frac{1}{2}}(t) = -\sigma k \Delta x_j(t) \left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right), \\ \\ \Delta m_j e_j'(t) - q_{j+\frac{1}{2}}(t) b_{j+\frac{1}{2}}(t) + q_{j-\frac{1}{2}}(t) b_{j-\frac{1}{2}}(t) = 0. \end{array}\right. \tag{35}$$

Therefore

$$\Delta m_j T_j(t) s_j'(t) = \left(q_{j+\frac{1}{2}}(t) b_{j+\frac{1}{2}}(t) - q_{j-\frac{1}{2}}(t) b_{j-\frac{1}{2}}(t)\right)$$

$$-b_j(t)\left(q_{j+\frac{1}{2}}(t) - q_{j-\frac{1}{2}}(t)\right) - q_j(t)\left(b_{j+\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t)\right)$$

$$-\sigma k \Delta x_j(t)\left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right) b_j(t)$$

$$= \left(q_{j+\frac{1}{2}}(t)b_{j+\frac{1}{2}}(t) - b_j(t)q_{j+\frac{1}{2}}(t) - q_j(t)b_{j+\frac{1}{2}}(t) + q_j(t)b_j(t)\right)$$
$$- \left(q_{j-\frac{1}{2}}(t)b_{j-\frac{1}{2}}(t) - b_j(t)q_{j-\frac{1}{2}}(t) - q_j(t)b_{j-\frac{1}{2}}(t) + q_j(t)b_j(t)\right)$$
$$-\sigma k\Delta x_j(t)\left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right)b_j(t)$$
$$= \left(q_{j+\frac{1}{2}}(t) - q_j(t)\right)\left(b_{j+\frac{1}{2}}(t) - b_j(t)\right)$$
$$- \left(q_{j-\frac{1}{2}}(t) - q_j(t)\right)\left(b_{j-\frac{1}{2}}(t) - b_j(t)\right)$$
$$-\sigma k\Delta x_j(t)\left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right)b_j(t).$$

Now we split the source term

$$-\sigma k\Delta x_j(t)\left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right)b_j(t) = -2\sigma k\Delta x_j(t)b_j(t)^2$$
$$-\sigma k\Delta x_j(t)\left(b_{j+\frac{1}{2}}(t) - b_j(t)\right)b_j(t) - \sigma k\Delta x_j(t)\left(b_{j-\frac{1}{2}}(t) - b_j(t)\right)b_j(t)$$

and we incorporate it in our last expression for $s_j'(t)$. One gets

$$\Delta m_j T_j(t)s_j'(t) = \left(q_{j+\frac{1}{2}}(t) - q_j(t) - \sigma k\Delta x_j(t)b_j(t)\right)\left(b_{j+\frac{1}{2}}(t) - b_j(t)\right)$$
$$- \left(q_{j-\frac{1}{2}}(t) - q_j(t) + \sigma k\Delta x_j(t)b_j(t)\right)\left(b_{j-\frac{1}{2}}(t) - b_j(t)\right) - 2\sigma k\Delta x_j(t)b_j(t)^2.$$

By definition of the fluxes (32)

$$q_{j+\frac{1}{2}}(t) - q_j(t) - \sigma k\Delta x_j(t)b_j(t) = q_{j+\frac{1}{2}}(t) - q_j(t) - \frac{\sigma}{2}\Delta x_j\rho_j v_j$$

and $b_{j+\frac{1}{2}}(t) - b_j(t)$ have the same sign. Their product is non negative. Similarly $q_{j-\frac{1}{2}}(t) - q_j(t) + \sigma k\Delta x_j(t)b_j(t)$ and $b_{j-\frac{1}{2}}(t) - b_j(t)$ have the opposite sign. Their product is non positive. The last term $-2\sigma k\Delta x_j(t)b_j(t)^2$ is non negative si k is non negative (and of course $\Delta x_j > 0$). In conclusion we have proved

Lemma 6. *The semi-discrete lagrangian scheme (35) with source term (28-29) is entropic*

$$s_j'(t) \geq 0.$$

Let us discuss the consequences of this property on the maximum principle. One has the formula

$$\Delta m_j s_j = \Delta x_j S_j = \left[\Delta x_j \frac{4}{3}\left(\frac{3}{3+|b_j|^2}\right)^{\frac{3}{4}}\right] E_j^{\frac{3}{4}}\left(1-|b_j|^2\right)^{\frac{1}{4}}.$$

Assume for simplicity the energy in the cell is positive and $|b| < 1$ at $t = 0$. Then $s_j(0) > 0$. So $s_j(t) > 0$. Therefore the product

$$\Delta x_j E_j^{\frac{3}{4}}\left(1-|b_j|^2\right)^{\frac{1}{4}} > 0 \tag{36}$$

is positive. We claim the maximum principle can be seen as a consequence of this inequality.

If the energy E_j is positive and the cell is non degenerate $0 < \Delta x_j < \infty$ then $|b_j| < 1$. By continuity the energy can not vanish. The only case where $|b| = 1$ is possible is if the mesh degenerates $\Delta x_j = \infty$. This is not possible in finite time since the size of the cell is a continuous function of the interface velocities $u_{j+\frac{1}{2}} = b^*_{j+\frac{1}{2}}$ and $u_{j-\frac{1}{2}} = b^*_{j-\frac{1}{2}}$. By inspection of formula (28) $b^*_{j+\frac{1}{2}}$ is finite which means the mesh is non degenerate for a sufficiently but positive time. One can also argue that $\Delta x_j = 0$ is not possible in finite time due to (36).

We can also analyze the semi-discrete scheme by means of the Cauchy-Lipshitz theorem for ordinary differential equations. The Cauchy-Lipshitz theorem states there exists a unique solution until a maximal positive time $T > 0$ is reached. The condition for the Cauchy-Lipshitz theorem to be true is the differentiability of the equation $X'(t) = F(X)$. This is true in our case provided the energy is non zero and the mesh is non degenerate. During this interval $[0, T[$ the above argument is true.

In view of completely discrete explicit methods it is worthwhile to estimate the maximal time T. Note that $b^*_{j+\frac{1}{2}}$ is a mean value of b_j and b_{j+1} plus a difference. Take $\sigma = 0$ for simplicity

$$b^* = \frac{\frac{E_L}{3+|b_L|^2} b_L + \frac{E_R}{3+|b_R|^2} b_R}{\frac{E_L}{3+|b_L|^2} + \frac{E_R}{3+|b_R|^2}} + \frac{\sqrt{3}}{4} \frac{q_R - q_L}{\frac{E_L}{3+|b_L|^2} + \frac{E_R}{3+|b_R|^2}}.$$

Therefore $|b^*| \leq 1$ and $T \geq \frac{\Delta x_j(0)}{2}$ in first approximation.

In view of completely discrete implicit methods, which is not the subject of this work, we notice inequality (36) will be true for any implicit schemes (provided a convenient Newton algorithm is used to compute the solution of the implicit system). It means the stability of the implicit method will be reached without any restriction on the time step.

6.2 Maximum principle for the Eulerian system

Let now turn to the analysis of the semi-discrete eulerian scheme. This semi-discrete scheme is (38) plus the projection fluxes written on a fixed mesh

$$\left\{ \begin{array}{l} \Delta x \rho_j'(t) - \frac{1}{\varepsilon}\left(b_{j+\frac{1}{2}}(t)\rho_{j+\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t)\rho_{j-\frac{1}{2}}(t)\right) = 0, \\ \Delta x(\rho_j v_j)'(t) + \frac{1}{\varepsilon}\Big(q_{j+\frac{1}{2}}(t) - b_{j+\frac{1}{2}}(t)\rho_{j+\frac{1}{2}}(t)v_{j+\frac{1}{2}}(t) \\ \qquad -q_{j-\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\rho_{j-\frac{1}{2}}(t)v_{j-\frac{1}{2}}(t)\Big) = -\frac{\sigma k \Delta x_j(t)}{\varepsilon^2}\left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right), \\ \\ \Delta x(\rho_j e_j)'(t) - \frac{1}{\varepsilon}\Big(q_{j+\frac{1}{2}}(t) b_{j+\frac{1}{2}}(t) + b_{j+\frac{1}{2}}(t)\rho_{j+\frac{1}{2}}(t)e_{j+\frac{1}{2}}(t) \\ \qquad\qquad -q_{j-\frac{1}{2}}(t) b_{j-\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t)\rho_{j-\frac{1}{2}}(t)e_{j-\frac{1}{2}}(t)\Big) = 0. \end{array} \right. \tag{37}$$

The projection fluxes are $\rho_{j+\frac{1}{2}}$, $\rho_{j+\frac{1}{2}}v_{j+\frac{1}{2}}$ and $\rho_{j+\frac{1}{2}}e_{j+\frac{1}{2}}$. They are upwinded accordingly to the sign of the interface velocity $u_{j+\frac{1}{2}} = -b_{j+\frac{1}{2}}$. Set $h = \rho$, $h = \rho v$ or $h = \rho e$. Then

$$h_{j+\frac{1}{2}} = \frac{1 + \text{sign}\left(u_{j+\frac{1}{2}}\right)}{2} h_j + \frac{1 - \text{sign}\left(u_{j+\frac{1}{2}}\right)}{2} h_{j+1}.$$

Lemma 7. *The eulerian semi-discrete scheme is entropic in the following sense. Assume the density and entropy are positive in every cell at $t = 0$, that is $\rho_j(t) > 0$ and $s_j(0) > 0$. Then $s_j(t) > 0$ for all time $t > 0$ and all cell j.*

We give a sketch of the proof. The eulerian semi-discrete scheme may be seen as the limit $\Delta t \to 0$ on a given mesh of the fully discrete lagrangian plus projection scheme. The fully discrete lagrangian scheme is entropy increasing under CFL. A first order projection is equivalent to a convex combination. Therefore the entropy after the projection is greater than a convex combination of entropies of neighboring cells. At the limit, one gets

$$\Delta x(\rho_j s_j)'(t) - \frac{1}{\varepsilon}\left(b_{j+\frac{1}{2}}(t)\rho_{j+\frac{1}{2}}(t)s_{j+\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t)\rho_{j-\frac{1}{2}}(t)s_{j-\frac{1}{2}}(t)\right) \geq 0.$$

This equation is similar to density equation plus a non negative source right hand side, except that the density ρ is replaced by the product ρs. Therefore ρs inherits all the properties of the classical solution of the equation of density. In particular the discrete density remains positive. Therefore it is also the case for the entropy $\rho_j(t)s_j(t)$. So $s_j(t) = \frac{\rho_j(t)s_j(t)}{\rho_j(t)} > 0$.

It is also possible to prove the result by a direct and tedious calculation of A where

$$A = \Delta x(\rho_j s_j)'(t) - \frac{1}{\varepsilon}\left(b_{j+\frac{1}{2}}(t)\rho_{j+\frac{1}{2}}(t)s_{j+\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t)\rho_{j-\frac{1}{2}}(t)s_{j-\frac{1}{2}}(t)\right).$$

The calculation is done by elimination of $(\rho s)'$ in function of ρ', $(\rho v)'$ and $(\rho e)'$. This is the method we have used for the analysis of the semi-discrete lagrangian scheme (35).

7 Diffusion limit

We analyze the diffusion limit of the semi-discrete eulerian scheme. The semi-discrete eulerian scheme is somehow the addition of the lagrangian step and the projection step. The semi-discrete lagrangian scheme where we have reintroduced the small parameter ε is

$$\begin{cases} \Delta m_j \tau_j'(t) + \frac{1}{\varepsilon}\left(b_{j+\frac{1}{2}}(t) - b_{j-\frac{1}{2}}(t)\right) = 0, \\ \Delta m_j v_j'(t) + \frac{1}{\varepsilon}\left(q_{j+\frac{1}{2}}(t) - q_{j-\frac{1}{2}}(t)\right) = -\frac{\sigma k \Delta x_j(t)}{\varepsilon^2}\left(b_{j+\frac{1}{2}}(t) + b_{j-\frac{1}{2}}(t)\right), \\ \\ \Delta m_j e_j'(t) - \frac{1}{\varepsilon}\left(q_{j+\frac{1}{2}}(t)b_{j+\frac{1}{2}}(t) + q_{j-\frac{1}{2}}(t)b_{j-\frac{1}{2}}(t)\right) = 0. \end{cases} \quad (38)$$

plus the flux formulas (33) which are now

$$\left\{\begin{array}{l} b_{j+\frac12} = \dfrac{\frac{E_j}{3+|b_j|^2} b_j + \frac{E_{j+1}}{3+|b_{j+1}|^2} b_{j+1}}{\frac{E_j}{3+|b_j|^2} + \frac{E_{j+1}}{3+|b_{j+1}|^2}} + \dfrac{\sqrt3}{4} \dfrac{\left(q_{j+1} + \frac{\sigma}{2\varepsilon}\Delta x \rho_{j+1} v_{j+1}\right) - \left(q_j - \frac{\sigma}{2\varepsilon}\Delta x \rho_j v_j\right)}{\frac{E_j}{3+|b_j|^2} + \frac{E_{j+1}}{3+|b_{j+1}|^2}}, \\ q_{j+\frac12} = \dfrac{\frac{3+|b_j|^2}{E_j}\left(q_j - \frac{\sigma}{2\varepsilon}\Delta x \rho_j v_j\right) + \frac{3+|b_{j+1}|^2}{E_{j+1}}\left(q_{j+1} + \frac{\sigma}{2\varepsilon}\Delta x \rho_{j+1} v_{j+1}\right)}{\frac{3+|b_j|^2}{E_j} + \frac{3+|b_{j+1}|^2}{E_{j+1}}} + \dfrac{4}{\sqrt3} \dfrac{b_{j+1} - b_j}{\frac{3+|b_j|^2}{E_j} + \frac{3+|b_{j+1}|^2}{E_{j+1}}}. \end{array}\right. \tag{39}$$

We are interested in the diffusion limit of the system (37). Any quantity is expanded as a series in ε as in

$$h = h^0 + \varepsilon h^1 + \varepsilon^2 h^2 + O(\varepsilon^3).$$

If needed we shall assume that $h^{0,1,2}$ are smooth functions of the time and space variables. We begin with $b_{j+\frac12}$. One has $b^0_{j+\frac12} - b^0_{j-\frac12} = 0$ so $b^0_{j+\frac12} = C$ is a constant which does not depend upon j. Assume that, for simplicity of the analysis, the boundary condition is zero for this quantity. Then

$$b^0_{j+\frac12} = 0, \quad \forall j. \tag{40}$$

Considering the ε^{-1} terms in (37), one has

$$(\rho_j v_j)^0 = 0, \quad \forall j. \tag{41}$$

Plugging (40-41) in (37) we get

$$q^0_{j+1} + \frac{\sigma}{2}\Delta x (\rho_{j+1} v_{j+1})^1 - q^0_j + \frac{\sigma}{2}\Delta x (\rho_j v_j)^1 = 0 \tag{42}$$

that is

$$\frac{q^0_{j+1} - q^0_j}{\Delta x} = -\frac{\sigma}{2}(\rho_{j+1} v_{j+1})^1 - \frac{\sigma}{2}\Delta x (\rho_j v_j)^1. \tag{43}$$

This equation is the discrete counterpart of (16). Considering (2) and $F = \rho v$, (41) implies $b^0_j = 0$ and also v^0_j. We have used the hypothesis that $T^0_j \neq 0$ which corresponds to the interesting case with non zero radiative energy. We have also used $\rho^0_j \neq 0$ for a similar reason. Therefore

$$(\rho_j v_j)^1 = \rho^0_j v^1_j + \rho^1_j v^0_j = \rho^0_j v^1_j, \quad \forall j. \tag{44}$$

So one can rewrite (42) as

$$\frac{q^0_{j+1} - q^0_j}{\Delta x} = -\sigma \frac{\rho^0_{j+1} v^1_{j+1} + \rho^0_j v^1_j}{2}. \tag{45}$$

Using a similar algebra in the first order expansion of $b_{j+\frac12}$ (recall $b^0_{j+\frac12} = 0$) one gets

$$b^1_{j+\frac12} = \frac{\frac{E^0_j}{3+|b^0_j|^2} b^1_j + \frac{E^0_{j+1}}{3+|b^0_{j+1}|^2} b^1_{j+1}}{\frac{E^0_j}{3+|b^0_j|^2} + \frac{E^0_{j+1}}{3+|b^0_{j+1}|^2}}$$

$$+\frac{\sqrt{3}}{4}\frac{\left(q_{j+1}^1+\frac{\sigma}{2}\Delta x(\rho_{j+1}v_{j+1})^1\right)-\left(q_j^1-\frac{\sigma}{2}\Delta x(\rho_j v_j)^1\right)}{\frac{E_j^0}{3+|b_j^0|^2}+\frac{E_{j+1}^0}{3+|b_{j+1}^0|^2}}$$

that is

$$b_{j+\frac12}^1=\alpha_j^0 b_{j+1}^1+(1-\alpha_j^0)b_j^1+O(\Delta x),\quad \alpha_j^0=\frac{\frac{E_j^0}{3+|b_j^0|^2}}{\frac{E_j^0}{3+|b_j^0|^2}+\frac{E_{j+1}^0}{3+|b_{j+1}^0|^2}}\tag{46}$$

Similar calculations but for $q_{j+\frac12}^0$ yield

$$q_{j+\frac12}^0=\frac{\frac{3+|b_j^0|^2}{E_j^0}\left(q_j^0-\frac{\sigma}{2}\Delta x(\rho_j v_j)^1\right)+\frac{3+|b_{j+1}^0|^2}{E_{j+1}^0}\left(q_{j+1}^0+\frac{\sigma}{2\varepsilon}\Delta x\rho_{j+1}^0 v_{j+1}^0\right)}{\frac{3+|b_j^0|^2}{E_j^0}+\frac{3+|b_{j+1}^0|^2}{E_{j+1}^0}}$$

$$+\frac{4}{\sqrt{3}}\frac{b_{j+1}^0-b_j^0}{\frac{3+|b_j^0|^2}{E_j^0}+\frac{3+|b_{j+1}^0|^2}{E_{j+1}^0}}$$

that is after simplifications

$$q_{j+\frac12}^0=\alpha_j^0 q_j^0+\alpha_{j+1}^0 q_{j+1}^0+O(\Delta x).\tag{47}$$

It is now possible to analyze the flux in the eulerian energy equation of (37). One has

$$\left(q_{j+\frac12}b_{j+\frac12}+b_{j+\frac12}\rho_{j+\frac12}e_{j+\frac12}\right)^0=0$$

and

$$\left(q_{j+\frac12}b_{j+\frac12}+b_{j+\frac12}\rho_{j+\frac12}e_{j+\frac12}\right)^1=q_{j+\frac12}^0 b_{j+\frac12}^1+\rho_{j+\frac12}^0 e_{j+\frac12}^0 b_{j+\frac12}^1.$$

We evaluate the first contribution in the right hand side $q_{j+\frac12}^0 b_{j+\frac12}^1$ with the help of (46) and (47). One gets

$$q_{j+\frac12}^0 b_{j+\frac12}^1=\left(\alpha_j^0 q_j^0+\alpha_{j+1}^0 q_{j+1}^0+O(\Delta x)\right)\left(\alpha_j^0 b_{j+1}^1+(1-\alpha_j^0)b_j^1+O(\Delta x)\right)$$

that is

$$q_{j+\frac12}^0 b_{j+\frac12}^1=\frac12 q_j^0 b_j^1+\frac12 q_{j+1}^0 b_{j+1}^1+O(\Delta x).\tag{48}$$

The second contribution in the right hand side $\rho_{j+\frac12}^0 e_{j+\frac12}^0 b_{j+\frac12}^1$ is, with the same kind of calculations,

$$\rho_{j+\frac12}^0 e_{j+\frac12}^0 b_{j+\frac12}^1=\frac12\rho_j^0 e_j^0 b_j^1+\frac12\rho_{j+1}^0 e_{j+1}^0 b_{j+1}^1+O(\Delta x).\tag{49}$$

Therefore the total eulerian flux is

$$\left(q_{j+\frac12}b_{j+\frac12}+b_{j+\frac12}\rho_{j+\frac12}e_{j+\frac12}\right)^1=\frac12(q_j^0+\rho_j^0 e_j^0)b_j^1+\frac12(q_{j+1}^0+\rho_{j+1}^0 e_{j+1}^0)b_{j+1}^1+O(\Delta x)$$

that is

$$\left(q_{j+\frac{1}{2}} b_{j+\frac{1}{2}} + b_{j+\frac{1}{2}} \rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}}\right)^1 = -\frac{\rho_j^0 v_j^1 + \rho_{j+1}^0 v_{j+1}^1}{2} + O(\Delta x). \qquad (50)$$

We know the asymptotic value of the right hand side, see formula (45). Since the definition of q implies that $q^0 = \frac{1}{3}E^0$, then one gets

$$\left(q_{j+\frac{1}{2}} b_{j+\frac{1}{2}} + b_{j+\frac{1}{2}} \rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}}\right)^1 = \frac{E_{j+1}^0 - E_j^0}{\Delta x} + O_{j+\frac{1}{2}}(3\sigma \Delta x).$$

The $O_{j+\frac{1}{2}}(\Delta x)$ is the error which is a priori different from one interface $(j+\frac{1}{2})$ to another $(l+\frac{1}{2},\, l \neq j)$. Plugging in the energy equation one has

$$\Delta x \frac{d}{dt} E_j^0 - \left(\frac{E_{j+1}^0 - E_j^0}{3\sigma \Delta x} + O_{j+\frac{1}{2}}(\Delta x) - \frac{E_j^0 - E_{j-1}^0}{3\sigma \Delta x} - O_{j-\frac{1}{2}}(\Delta x) \right) = 0.$$

In summary we have proved

Lemma 8. *The asymptotic limit of the system (37) is the discrete diffusion equation*

$$\frac{d}{dt} E_j - \frac{E_{j+1} - 2E_j + E_{j-1}}{3\sigma \Delta x^2} = O_j^{weak}(\Delta x).$$

The right hand side is

$$O_j^{weak}(\Delta x) = \frac{O_{j+\frac{1}{2}}(\Delta x) - O_{j-\frac{1}{2}}(\Delta x)}{\Delta x}.$$

In the finite difference sense one has $O_j^{weak}(\Delta x) = O(1)$. But this term is consistent with the weak formulation of the heat equation because it is the difference of two $O(1)$ terms. That is

$$O^{weak}(\Delta x) = O(\Delta x) \text{ in the finite volume sense.}$$

In other words

$$O^{weak}(\Delta x) \to 0 \text{ as } \Delta x \to 0$$

in the weak sense.

In view of fully discrete schemes, one can notice that the splitting in time of a lagrangian plus projection scheme makes little difference in the analysis. A small change is the eulerian flux for the energy equation which is computed in a splitted fashion. It introduces a additional $O(\Delta t)$ in asymptotic expansion of the eulerian flux.

8 Numerical results

The algorithm used in the numerical test is the lagrangian plus projection explicit scheme with fluxes given by (33). The explicit scheme is enough to illustrate the correctness of the approach proposed in this work.

We give the results of three numerical experiments. The first one shows that the solution of the GDL formulation of the equations is independent of the definition of the density. The second one shows that the scheme captures an analytical solution in the streaming regime and preserves the flux-limited property. The last test case is representative of the diffusion limit of the scheme.

8.1 Test case 1: radiative Riemann problem

We consider a Riemann problem. The coefficients are $\sigma = 0$ everywhere and $\varepsilon = 1$. The initial values are

$$(E, F_1) = (1, 0) \text{ for } x < 0.5,$$

and

$$(E, F_1) = (0.1, 0) \text{ for } 0.5 < x.$$

The second component of the radiative flux is zero $F_2 \equiv 0$. The solution consists in a mathematical rarefaction fan on the left and a shock on the right. We observe in figure 1 a very good agreement. One can check, as claimed in section 5.4, that the results are independent of the initial value of the density ρ.

8.2 Test case 2: streaming regime

The coefficients are $\sigma = 0$ everywhere and $\varepsilon = 1$. The initial values are

$$(E, F_1, F_2) = (1, \frac{0.7 - x}{0.4}, \sqrt{1 - F_1^2}) \text{ for } 0.3 < x < 0.7,$$

and

$$(E, F_1, F_2) = (0, 0, 0) \text{ elsewhere.}$$

At $t < 0.4$ the analytical solution is

$$(E, F_1, F_2) = (\frac{0.4}{0.4 - t}, \frac{0.4(0.7 - x)}{(0.4 - t)^2}, \sqrt{1 - F_1^2}) \text{ for } 0.3 + t < x < 0.7,$$

and

$$(E, F_1, F_2) = (0, 0, 0) \text{ elsewhere.}$$

This analytical solution is also the analytical solution of the equation of transport for a prepared data, see proposition 2. In practice we have used this equivalence to compute this analytical solution. At time $t = 0.4$ the solution is a measure. For $t > 0.4$ the code still computes a numerical solution, but a priori this solution is not a solution of the transport equation.

We observe the solution at time $t = 0.2$ is in good agreement with the analytical solution. The curves with 4000 cells (figure 3) are closer to the analytical solution than the curves with 400 cells (figure 2). The ratio of the energy flux over the radiation energy is bounded by one, see figure 4. In figure 5 we show the numerical solution at the singular time $t = 0.04$: it is approximatively a Dirac function at $x = 0.7$.

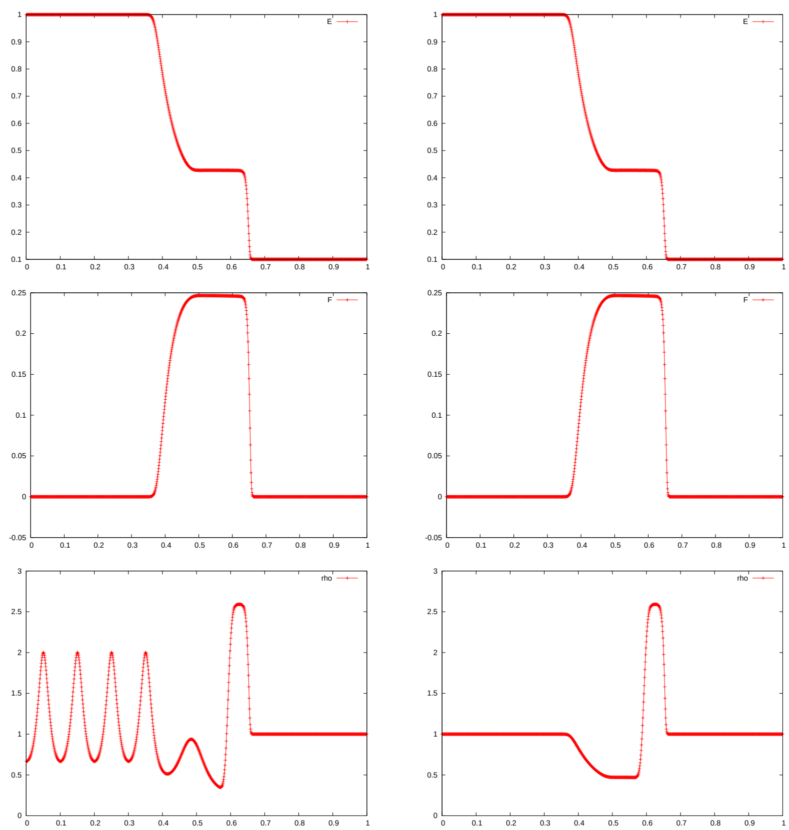

Figure 1: E, F and ρ at $t = 0.2$. Results in the first and second columns have been computed with different densities but same radiative energy and radiative flux

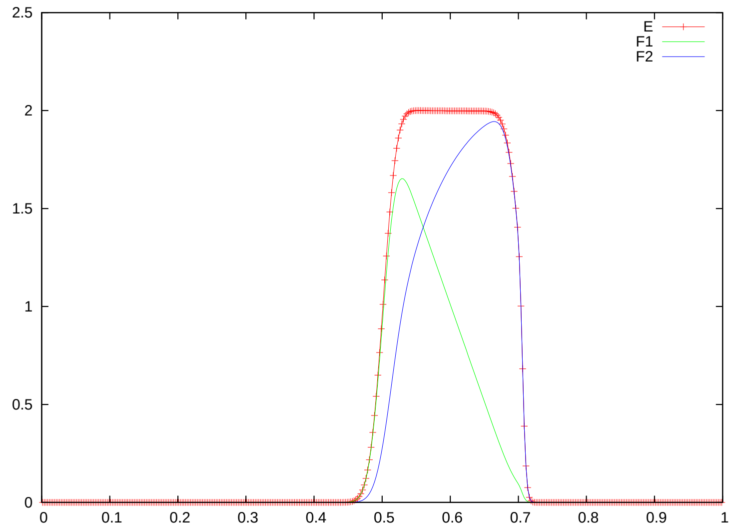

Figure 2: E, F_1 and F_2 at time $t = 0.2$. 400 cells

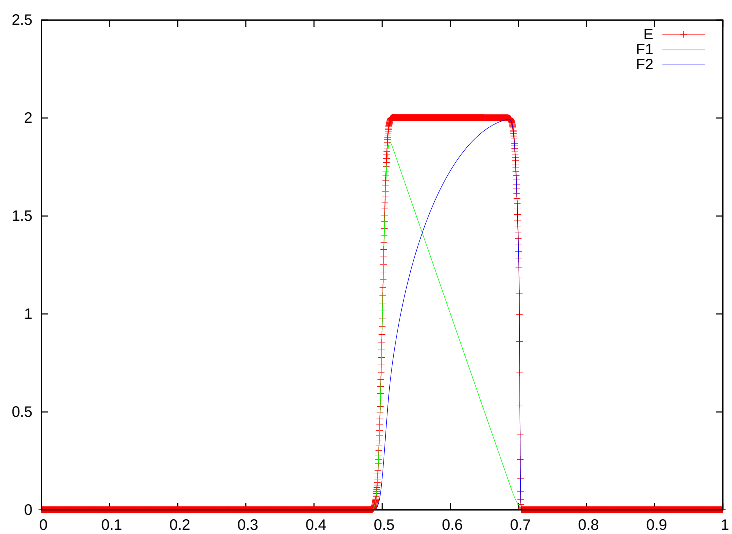

Figure 3: E, F_1 and F_2 at time $t = 0.2$. 4000 cells

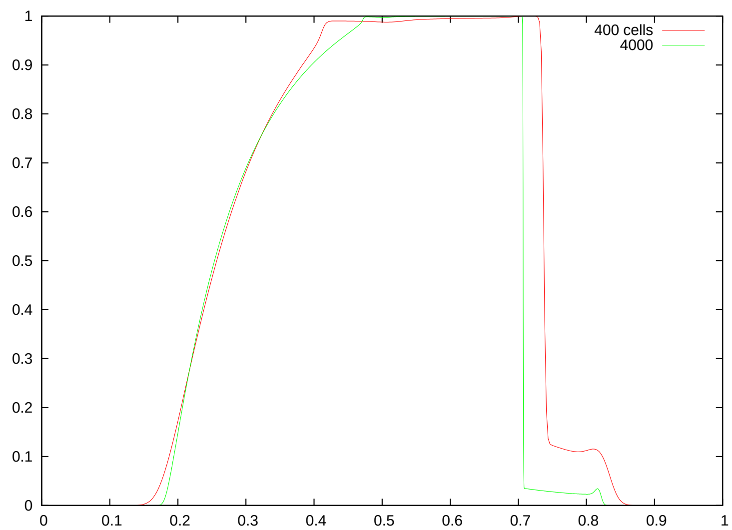

Figure 4: $f = \frac{\sqrt{F_1^2+F_2^2}}{E}$. 400 and 4000 cells

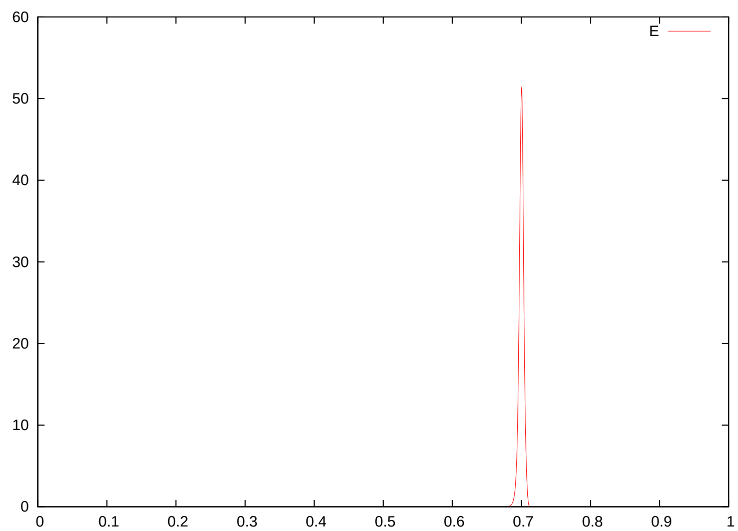

Figure 5: Radiative energy at time $t = 0.4$. 4000 cells. The numerical profile is an approximation of the dirac function $x \mapsto 0.4\ \delta_{0.7}(x)$

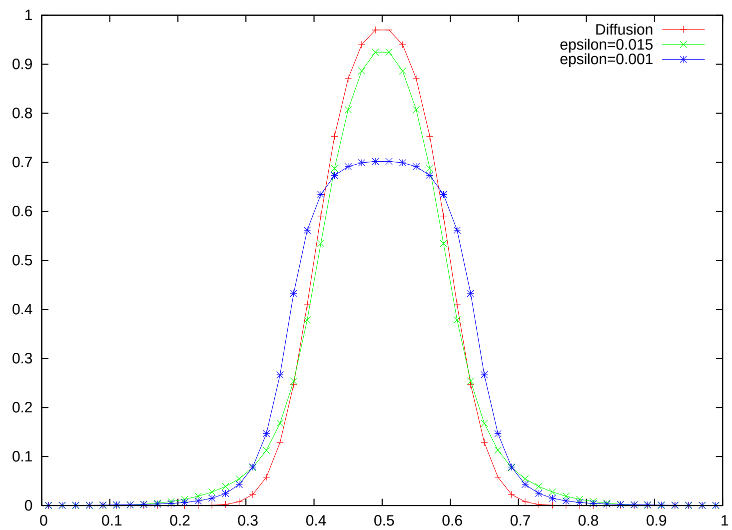

Figure 6: Classical fluxes. Non convergence towards the solution of the heat equation as $\varepsilon \to 0$. The curve for $\varepsilon = 0.015$ is completely different from the solution of the diffusion (heat) equation. Final time $T = 0.003$

8.3 Test case 3: diffusion limit

The coefficients are $\sigma = 1$ everywhere and $10^{-4} \leq \varepsilon \leq 10^{-1}$. The initial values are

$$(E, F_1, F_2) = (1, 0, 0) \text{ for } 0.4 < x < 0.6, \quad (E, F_1, F_2) = (10^{-6}, 0, 0) \text{ elsewhere.}$$

We first show in figure 6 what happens with the classical fluxes (28). In this case the diffusion limit is not captured and the solver becomes pathological as $\varepsilon \to 0$.

In figure 7 we plot four curves computed on a coarse grid of 50 cells. One is the solution of the heat equation and is the reference solution. The three others are computed with the moment model for decreasing ε, that is 0.1, 0.05 and 0.015. The stability of the algorithm is evident. The convergence of the coarse grid discrete solution towards the coarse grid solution of the heat equation is achieved.

In table 1, we show the relative error (in the L^∞ norm) between the discrete solution of the heat equation and the discrete solution of the moment model. This error is made of two contributions: one contribution is the model error, the other one is the discretization error. In first and second columns the model error is dominant and this is why the error increases as Δx tends to zero. In the fourth and fifth columns, the discretization error is dominant so the error decreases as Δx tends to zero. The third column is somehow in between, the model error is of the same order than the discretization error. The behavior on lines is monotone and illustrates the result of convergence of lemma 8 for $\varepsilon \to 0$.

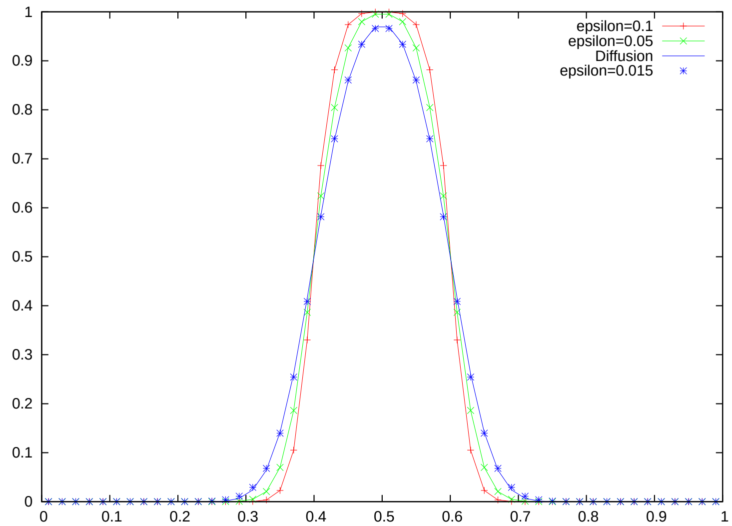

Figure 7: New fluxes. Convergence study towards the solution of the heat equation as $\varepsilon \to 0$. Final time $T = 0.003$. The curve for $\varepsilon = 0.015$ is not distinguishable from the solution of the diffusion (heat) equation. See table 1 for a quantitative study of convergence

	$\varepsilon = 10^{-1}$	$\varepsilon = 5.10^{-2}$	$\varepsilon = 1.5\ 10^{-2}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$
$\Delta x = 1/50$	0.15	0.061	0.012	0.017	0.025
$\Delta x = 1/100$	0.17	0.080	0.009	0.010	0.014
$\Delta x = 1/200$	0.20	0.106	0.009	0.066	0.008
$\Delta x = 1/400$	0.24	0.130	0.012	0.004	0.004

Table 1: Relative L^∞ errors between the discrete solution of the heat equation and the discrete solution of the moment model (parameter ε) computed on the same grid (parameter Δx). Final time $T = 0.003$

8.4 Conclusion

Numerical results validate the theoretical developments. In particular the new scheme is independent of the artificial density used in the calculation, the streaming regime for smooth solutions and the diffusion regime are well captured. We think the Lagrange plus projection scheme can be used for other GDL systems.

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